

## SUPPLEMENTARY MATERIAL

### CHAPTER 3

#### 3.6 Proofs and Simple Applications of Sine and Cosine Formulae

Let  $ABC$  be a triangle. By angle  $A$ , we mean the angle between the sides  $AB$  and  $AC$  which lies between  $0^\circ$  and  $180^\circ$ . The angles  $B$  and  $C$  are similarly defined. The sides  $AB$ ,  $BC$  and  $CA$  opposite to the vertices  $C$ ,  $A$  and  $B$  will be denoted by  $c$ ,  $a$  and  $b$  respectively (see Fig. 3.15).

**Theorem 1** (Sine formulae) In any triangle, sides are proportional to the sines of the opposite angles. That is, in a triangle  $ABC$

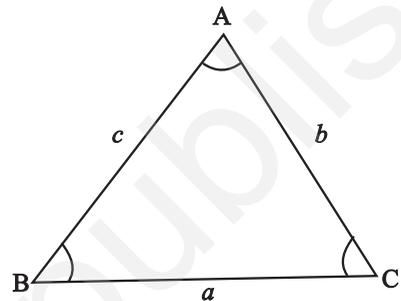
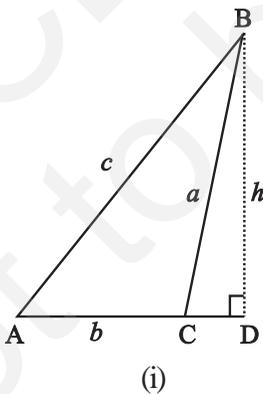


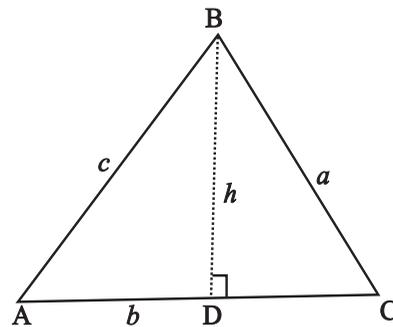
Fig. 3.15

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

**Proof** Let  $ABC$  be either of the triangles as shown in Fig. 3.16 (i) and (ii).



(i)



(ii)

Fig. 3.16

The altitude  $h$  is drawn from the vertex  $B$  to meet the side  $AC$  in point  $D$  [in (i)  $AC$  is

produced to meet the altitude in D]. From the right angled triangle ABD in Fig. 3.16(i), we have

$$\sin A = \frac{h}{c}, \text{ i.e., } h = c \sin A \tag{1}$$

and  $\sin (180^\circ - C) = \frac{h}{a} \Rightarrow h = a \sin C$  (2)

From (1) and (2), we get

$$c \sin A = a \sin C, \text{ i.e., } \frac{\sin A}{a} = \frac{\sin C}{c} \tag{3}$$

Similarly, we can prove that

$$\frac{\sin A}{a} = \frac{\sin B}{b} \tag{4}$$

From (3) and (4), we get

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

For triangle ABC in Fig. 3.16 (ii), equations (3) and (4) follow similarly.

**Theorem 2** (Cosine formulae) Let A, B and C be angles of a triangle and  $a, b$  and  $c$  be lengths of sides opposite to angles A, B and C respectively, then

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = c^2 + a^2 - 2ca \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

**Proof** Let ABC be triangle as given in Fig. 3.17 (i) and (ii)

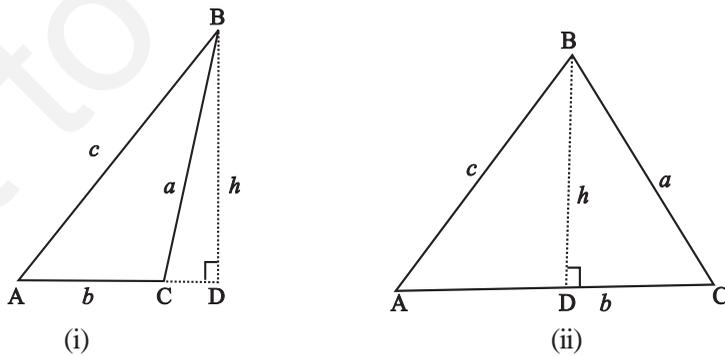


Fig. 3.17

Referring to Fig. 3.17 (ii), we have

$$\begin{aligned} BC^2 &= BD^2 + DC^2 = BD^2 + (AC - AD)^2 \\ &= BD^2 + AD^2 + AC^2 - 2AC \cdot AD \\ &= AB^2 + AC^2 - 2AC \cdot AB \cos A \end{aligned}$$

$$\text{or } a^2 = b^2 + c^2 - 2bc \cos A$$

Similarly, we can obtain

$$b^2 = c^2 + a^2 - 2ca \cos B$$

$$\text{and } c^2 = a^2 + b^2 - 2ab \cos C$$

Same equations can be obtained for Fig. 3.17 (i), where C is obtuse.

A convenient form of the cosine formulae, when angles are to be found are as follows:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

**Example 25** In triangle ABC, prove that

$$\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}$$

$$\tan \frac{C - A}{2} = \frac{c - a}{c + a} \cot \frac{B}{2}$$

$$\tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2}$$

**Proof** By sine formulae, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k(\text{say}).$$

Therefore, 
$$\frac{b - c}{b + c} = \frac{k(\sin B - \sin C)}{k(\sin B + \sin C)}$$

$$= \frac{2 \cos \frac{B + C}{2} \sin \frac{B - C}{2}}{2 \sin \frac{B + C}{2} \cos \frac{B - C}{2}}$$

$$= \cot \frac{(B + C)}{2} \tan \frac{(B - C)}{2}$$

$$= \cot \left( \frac{\pi}{2} - \frac{A}{2} \right) \tan \left( \frac{B - C}{2} \right)$$

$$= \frac{\tan \frac{B - C}{2}}{\cot \frac{A}{2}}$$

Therefore, 
$$\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}$$

Similarly, we can prove other results. These results are well known as Napier’s Analogies.

**Example 26** In any triangle ABC, prove that

$$a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0$$

**Solution** Consider

$$a \sin (B - C) = a [\sin B \cos C - \cos B \sin C] \tag{1}$$

Now 
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (say)}$$

Therefore,  $\sin A = ak, \sin B = bk, \sin C = ck$

Substituting the values of  $\sin B$  and  $\sin C$  in (1) and using cosine formulae, we get

$$\begin{aligned} a \sin(B - C) &= a \left[ bk \left( \frac{a^2 + b^2 - c^2}{2ab} \right) - ck \left( \frac{c^2 + a^2 - b^2}{2ac} \right) \right] \\ &= \frac{k}{2} (a^2 + b^2 - c^2 - c^2 - a^2 + b^2) \\ &= k(b^2 - c^2) \end{aligned}$$

Similarly,  $b \sin(C - A) = k(c^2 - a^2)$

and  $c \sin(A - B) = k(a^2 - b^2)$

$$\begin{aligned} \text{Hence L.H.S} &= k(b^2 - c^2 + c^2 - a^2 + a^2 - b^2) \\ &= 0 = \text{R.H.S.} \end{aligned}$$

**Example 27** The angle of elevation of the top point P of the vertical tower PQ of height  $h$  from a point A is  $45^\circ$  and from a point B, the angle of elevation is  $60^\circ$ , where B is a point at a distance  $d$  from the point A measured along the line AB which makes an angle  $30^\circ$  with AQ. Prove that  $d = h(\sqrt{3} - 1)$

**Proof** From the Fig. 3.18, we have  $\angle PAQ = 45^\circ$ ,  $\angle BAQ = 30^\circ$ ,  $\angle PBH = 60^\circ$

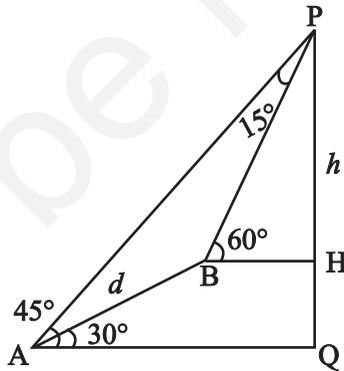


Fig. 3.18

Clearly

$$\angle APQ = 45^\circ, \angle BPH = 30^\circ, \text{ giving } \angle APB = 15^\circ$$

Again

$$\angle PAB = 15^\circ \Rightarrow \angle ABP = 150^\circ$$

From triangle APQ, we have  $AP^2 = h^2 + h^2 = 2h^2$  (Why?)

or  $AP = \sqrt{2}h$

Applying sine formulae in  $\Delta ABP$ , we get

$$\frac{AB}{\sin 15^\circ} = \frac{AP}{\sin 150^\circ} \Rightarrow \frac{d}{\sin 15^\circ} = \frac{\sqrt{2}h}{\sin 150^\circ}$$

i.e.,  $d = \frac{\sqrt{2}h \sin 15^\circ}{\sin 30^\circ}$   
 $= h(\sqrt{3} - 1)$  (why?)

**Example 28** A lamp post is situated at the middle point M of the side AC of a triangular plot ABC with BC = 7 m, CA = 8 m and AB = 9 m. Lamp post subtends an angle  $15^\circ$  at the point B. Determine the height of the lamp post.

**Solution** From the Fig. 3.19, we have AB = 9 = c, BC = 7 m = a and AC = 8 m = b.

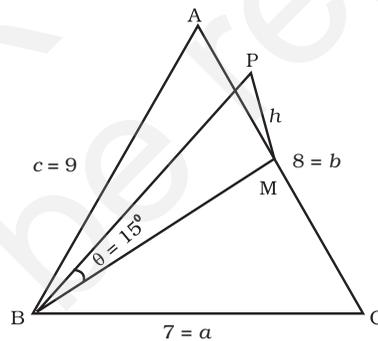


Fig. 3.19

M is the mid-point of the side AC at which lamp post MP of height  $h$  (say) is located. Again, it is given that lamp post subtends an angle  $\theta$  (say) at B which is  $15^\circ$ .

Applying cosine formulae in  $\Delta ABC$ , we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49 + 64 - 81}{2 \times 7 \times 8} = \frac{2}{7} \tag{1}$$

Similarly using cosine formulae in  $\triangle BMC$ , we get

$$BM^2 = BC^2 + CM^2 - 2 BC \times CM \cos C.$$

Here  $CM = \frac{1}{2} CA = 4$ , since M is the mid-point of AC.

Therefore, using (1), we get

$$\begin{aligned} BM^2 &= 49 + 16 - 2 \times 7 \times 4 \times \frac{2}{7} \\ &= 49 \end{aligned}$$

$$\text{or } BM = 7$$

Thus, from  $\triangle BMP$  right angled at M, we have

$$\tan \theta = \frac{PM}{BM} = \frac{h}{7}$$

$$\text{or } \frac{h}{7} = \tan(15^\circ) = 2 - \sqrt{3} \quad (\text{why?})$$

$$\text{or } h = 7(2 - \sqrt{3}) \text{ m.}$$

### Exercise 3.5

In any triangle ABC, if  $a = 18$ ,  $b = 24$ ,  $c = 30$ , find

1.  $\cos A$ ,  $\cos B$ ,  $\cos C$

(Ans.  $\frac{4}{5}, \frac{3}{5}, 0$ )

2.  $\sin A$ ,  $\sin B$ ,  $\sin C$

(Ans.  $\frac{3}{5}, \frac{4}{5}, 1$ )

For any triangle ABC, prove that

$$3. \frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$$

4. 
$$\frac{a-b}{c} = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}}$$
5. 
$$\sin\frac{B-C}{2} = \frac{b-c}{a} \cos\frac{A}{2}$$
6. 
$$a(b \cos C - c \cos B) = b^2 - c^2$$
7. 
$$a(\cos C - \cos B) = 2(b-c) \cos^2\frac{A}{2}$$
8. 
$$\frac{\sin(B-C)}{\sin(B+C)} = \frac{b^2 - c^2}{a^2}$$
9. 
$$(b+c) \cos\frac{B+C}{2} = a \cos\frac{B-C}{2}$$
10. 
$$a \cos A + b \cos B + c \cos C = 2a \sin B \sin C$$
11. 
$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$
12. 
$$(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$$
13. 
$$\frac{b^2 - c^2}{a^2} \sin 2A + \frac{c^2 - a^2}{b^2} \sin 2B + \frac{a^2 - b^2}{c^2} \sin 2C = 0$$
14. A tree stands vertically on a hill side which makes an angle of  $15^\circ$  with the horizontal. From a point on the ground 35 m down the hill from the base of the tree, the angle of elevation of the top of the tree is  $60^\circ$ . Find the height of the tree. (Ans.  $35\sqrt{2}m$ )
15. Two ships leave a port at the same time. One goes 24 km per hour in the direction  $N45^\circ E$  and other travels 32 km per hour in the direction  $S75^\circ E$ . Find the distance between the ships at the end of 3 hours. (Ans. 86.4 km (approx.))
16. Two trees, A and B are on the same side of a river. From a point C in the river the distance of the trees A and B is 250 m and 300 m, respectively. If the angle C is  $45^\circ$ , find the distance between the trees (use  $\sqrt{2} = 1.414$ ). (Ans. 215.5 m)

## CHAPTER 5

## 5.7 Square-root of a Complex Number

We have discussed solving of quadratic equations involving complex roots on page 108-109 of the textbook. Here we explain the particular procedure for finding square root of a complex number expressed in the standard form. We illustrate the same by an example.

**Example 12** Find the square root of  $-7 - 24i$

**Solution** Let  $x + iy = \sqrt{-7 - 24i}$

$$\text{Then } (x + iy)^2 = -7 - 24i$$

$$\text{or } x^2 - y^2 + 2xyi = -7 - 24i$$

Equating real and imaginary parts, we have

$$x^2 - y^2 = -7 \quad (1)$$

$$2xy = -24$$

We know the identity

$$x^2 + y^2 + x^2 - y^2 + (2xy)^2$$

$$= 49 + 576$$

$$= 625$$

$$\text{Thus, } x^2 + y^2 = 25 \quad (2)$$

From (1) and (2),  $x^2 = 9$  and  $y^2 = 16$

$$\text{or } x = \pm 3 \text{ and } y = \pm 4$$

Since the product  $xy$  is negative, we have

$$x = 3, y = -4 \text{ or } x = -3, y = 4$$

Thus, the square roots of  $-7 - 24i$  are  $3 - 4i$  and  $-3 + 4i$ .

## Exercise 5.4

Find the square roots of the following:

1.  $-15 - 8i$  ( Ans.  $1 - 4i, -1 + 4i$  )

2.  $-8 - 6i$  ( Ans.  $1 - 3i, -1 + 3i$  )

3.  $1 - i$  (Ans.  $\left( \pm \sqrt{\frac{\sqrt{2} + 1}{2}} \mp \sqrt{\frac{\sqrt{2} - 1}{2}} i \right)$ )

4.  $-i$  (Ans.  $\left( \pm \frac{1}{\sqrt{2}} \mp \frac{1}{\sqrt{2}} i \right)$ )

5.  $i$  (Ans.  $\left( \pm \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i \right)$ )

6.  $1 + i$  (Ans.  $\left( \pm \sqrt{\frac{\sqrt{2} + 1}{2}} \pm \sqrt{\frac{\sqrt{2} - 1}{2}} i \right)$ )

### CHAPTER 9

#### 9.7 Infinite G.P. and its Sum

G.P. of the form  $a, ar, ar^2, ar^3, \dots$  is called infinite G.P. Now, to find the formulae for finding sum to infinity of a G.P., we begin with an example.

Let us consider the G.P.,

$$1, \frac{2}{3}, \frac{4}{9}, \dots$$

Here  $a = 1, r = \frac{2}{3}$ . We have

$$S_n = \frac{1 - \left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}} = 3 \left[ 1 - \left(\frac{2}{3}\right)^n \right]$$

Let us study the behaviour of  $\left(\frac{2}{3}\right)^n$  as  $n$  becomes larger and larger:

$n$	1	5	10	20
$\left(\frac{2}{3}\right)^n$	0.6667	0.1316872428	0.01734152992	0.00030072866

We observe that as  $n$  becomes larger and larger,  $\left(\frac{2}{3}\right)^n$  becomes closer and closer to zero. Mathematically, we say that as  $n$  becomes sufficiently large,  $\left(\frac{2}{3}\right)^n$  becomes sufficiently small. In other words as  $n \rightarrow \infty$ ,  $\left(\frac{2}{3}\right)^n \rightarrow 0$ . Consequently, we find that the sum of infinitely many terms is given by  $S_\infty = 3$ .

Now, for a geometric progression,  $a, ar, ar^2, \dots$ , if numerical value of common ratio  $r$  is less than 1, then

$$S_n = \frac{a(1-r^n)}{(1-r)} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

In this case as  $n \rightarrow \infty$ ,  $r^n \rightarrow 0$  since  $|r| < 1$ . Therefore

$$S_n \rightarrow \frac{a}{1-r}$$

Symbolically sum to infinity is denoted by  $S_\infty$  or  $S$ .

Thus, we have  $S = \frac{a}{1-r}$ .

For examples,

$$(i) \quad 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = \frac{1}{1 - \frac{1}{2}} = 2.$$

$$(ii) \quad 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots = \frac{1}{1 - \left(\frac{-1}{2}\right)} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$$

### Exercise 9.4

Find the sum to infinity in each of the following Geometric Progression.

1.  $1, \frac{1}{3}, \frac{1}{9}, \dots$  (Ans. 1.5)      2.  $6, 1.2, .24, \dots$  (Ans. 7.5)

3.  $5, \frac{20}{7}, \frac{80}{49}, \dots$  (Ans.  $\frac{35}{3}$ )      4.  $\frac{-3}{4}, \frac{3}{16}, \frac{-3}{64}, \dots$  (Ans.  $\frac{-3}{5}$ )

5. Prove that  $3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \dots = 3$
6. Let  $x = 1 + a + a^2 + \dots$  and  $y = 1 + b + b^2 + \dots$ , where  $|a| < 1$  and  $|b| < 1$ . Prove that

$$1 + ab + a^2b^2 + \dots = \frac{xy}{x + y - 1}$$

## CHAPTER 10

### 10.6 Equation of Family of Lines Passing Through the Point of Intersection of Two Lines

Let the two intersecting lines  $l_1$  and  $l_2$  be given by

$$A_1x + B_1y + C_1 = 0 \tag{1}$$

and  $A_2x + B_2y + C_2 = 0 \tag{2}$

From the equations (1) and (2), we can form an equation

$$A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0 \tag{3}$$

where  $k$  is an arbitrary constant called parameter. For any value of  $k$ , the equation (3) is of first degree in  $x$  and  $y$ . Hence it represents a family of lines. A particular member of this family can be obtained for some value of  $k$ . This value of  $k$  may be obtained from other conditions.

**Example 20** Find the equation of line parallel to the  $y$ -axis and drawn through the point of intersection of  $x - 7y + 5 = 0$  and  $3x + y - 7 = 0$

**Solution** The equation of any line through the point of intersection of the given lines is of the form

$$x - 7y + 5 + k(3x + y - 7) = 0$$

i.e.,  $(1 + 3k)x + (k - 7)y + 5 - 7k = 0 \tag{1}$

If this line is parallel to  $y$ -axis, then the coefficient of  $y$  should be zero, i.e.,  $k - 7 = 0$  which gives  $k = 7$ .

Substituting this value of  $k$  in the equation (1), we get

$$22x - 44 = 0, \text{ i.e., } x - 2 = 0, \text{ which is the required equation.}$$

### Exercise 10.4

- Find the equation of the line through the intersection of lines  $3x + 4y = 7$  and  $x - y + 2 = 0$  and whose slope is 5. (Ans.  $35x - 7y + 18 = 0$ )
- Find the equation of the line through the intersection of lines  $x + 2y - 3 = 0$  and  $4x - y + 7 = 0$  and which is parallel to  $5x + 4y - 20 = 0$  (Ans.  $15x + 12y - 7 = 0$ )
- Find the equation of the line through the intersection of the lines  $2x + 3y - 4 = 0$  and  $x - 5y = 7$  that has its  $x$ -intercept equal to  $-4$ . (Ans.  $10x + 93y + 40 = 0$ .)
- Find the equation of the line through the intersection of  $5x - 3y = 1$  and  $2x + 3y - 23 = 0$  and perpendicular to the line  $5x - 3y - 1 = 0$ . (Ans.  $63x + 105y - 781 = 0$ )

### 10.7 Shifting of Origin

An equation corresponding to a set of points with reference to a system of coordinate axes may be simplified by taking the set of points in some other suitable coordinate system such that all geometric properties remain unchanged. One such transformation is that in which the new axes are transformed parallel to the original axes and origin is shifted to a new point. A transformation of this kind is called a *translation of axes*.

The coordinates of each point of the plane are changed under a translation of axes. By knowing the relationship between the old coordinates and the new coordinates of points, we can study the analytical problem in terms of new system of coordinate axes.

To see how the coordinates of a point of the plane changed under a translation of axes, let us take a point  $P(x, y)$  referred to the axes  $OX$  and  $OY$ . Let  $O'X'$  and  $O'Y'$  be new axes parallel to  $OX$  and  $OY$  respectively, where  $O'$  is the new origin. Let  $(h, k)$

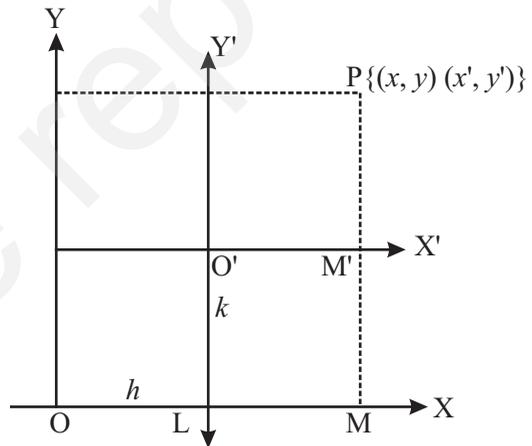


Fig. 10.21

be the coordinates of  $O'$  referred to the old axes, i.e.,  $OL = h$  and  $LO' = k$ . Also,  $OM = x$  and  $MP = y$  (see Fig.10.21)

Let  $O'M' = x'$  and  $M'P = y'$  be respectively, the abscissa and ordinates of a point P referred to the new axes  $O'X'$  and  $O'Y'$ . From Fig.10.21, it is easily seen that

$$OM = OL + LM, \text{ i.e., } x = h + x'$$

$$\text{and } MP = MM' + M'P, \text{ i.e., } y = k + y'$$

$$\text{Hence, } x = x' + h, y = y' + k$$

These formulae give the relations between the old and new coordinates.

**Example 21** Find the new coordinates of point  $(3, -4)$  if the origin is shifted to  $(1, 2)$  by a translation.

**Solution** The coordinates of the new origin are  $h = 1, k = 2$ , and the original coordinates are given to be  $x = 3, y = -4$ .

The transformation relation between the old coordinates  $(x, y)$  and the new coordinates  $(x', y')$  are given by

$$x = x' + h \quad \text{i.e.,} \quad x' = x - h$$

$$\text{and } y = y' + k \quad \text{i.e.,} \quad y' = y - k$$

Substituting the values, we have

$$x' = 3 - 1 = 2 \text{ and } y' = -4 - 2 = -6$$

Hence, the coordinates of the point  $(3, -4)$  in the new system are  $(2, -6)$ .

**Example 22** Find the transformed equation of the straight line  $2x - 3y + 5 = 0$ , when the origin is shifted to the point  $(3, -1)$  after translation of axes.

**Solution** Let coordinates of a point P changes from  $(x, y)$  to  $(x', y')$  in new coordinate axes whose origin has the coordinates  $h = 3, k = -1$ . Therefore, we can write the transformation formulae as  $x = x' + 3$  and  $y = y' - 1$ . Substituting, these values in the given equation of the straight line, we get

$$2(x' + 3) - 3(y' - 1) + 5 = 0$$

$$\text{or } 2x' - 3y' + 14 = 0$$

Therefore, the equation of the straight line in new system is  $2x - 3y + 14 = 0$

### Exercise 10.5

- Find the new coordinates of the points in each of the following cases if the origin is shifted to the point  $(-3, -2)$  by a translation of axes.
 

(i) $(1, 1)$	(Ans. $(4, 3)$ )	(ii) $(0, 1)$	(Ans. $(3, 3)$ )
(iii) $(5, 0)$	(Ans. $(8, 2)$ )	(iv) $(-1, -2)$	(Ans. $(2, 0)$ )
(v) $(3, -5)$	(Ans. $(6, -3)$ )		

2. Find what the following equations become when the origin is shifted to the point (1, 1)
- (i)  $x^2 + xy - 3y^2 - y + 2 = 0$  (Ans.  $x^2 - 3y^2 + xy + 3x - 6y + 1 = 0$ )
- (ii)  $xy - y^2 - x + y = 0$  (Ans.  $xy - y^2 = 0$ )
- (iii)  $xy - x - y + 1 = 0$  (Ans.  $xy = 0$ )

## CHAPTER 13

### 13.5 Limits Involving Exponential and Logarithmic Functions

Before discussing evaluation of limits of the expressions involving exponential and logarithmic functions, we introduce these two functions stating their domain, range and also sketch their graphs roughly.

Leonhard Euler (1707–1783), the great Swiss mathematician introduced the number  $e$  whose value lies between 2 and 3. This number is useful in defining exponential function and is defined as  $f(x) = e^x$ ,  $x \in \mathbf{R}$ . Its domain is  $\mathbf{R}$ , range is the set of positive real numbers. The graph of exponential function, i.e.,  $y = e^x$  is as given in Fig.13.11.

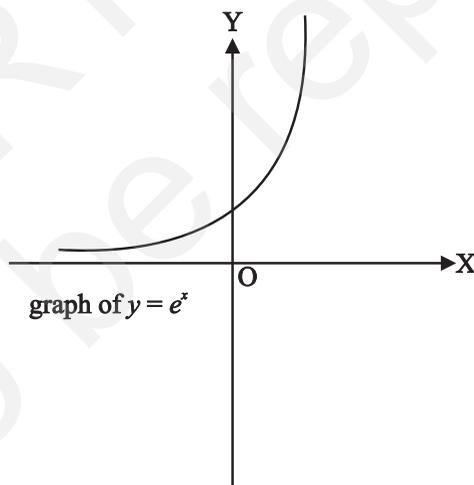


Fig. 13.11

Similarly, the logarithmic function expressed as  $\log_e \mathbf{R}^+ \rightarrow \mathbf{R}$  is given by  $\log_e x = y$ , if and only if  $e^y = x$ . Its domain is  $\mathbf{R}^+$  which is the set of all positive real numbers and range is  $\mathbf{R}$ . The graph of logarithmic function  $y = \log_e x$  is shown in Fig.13.12.

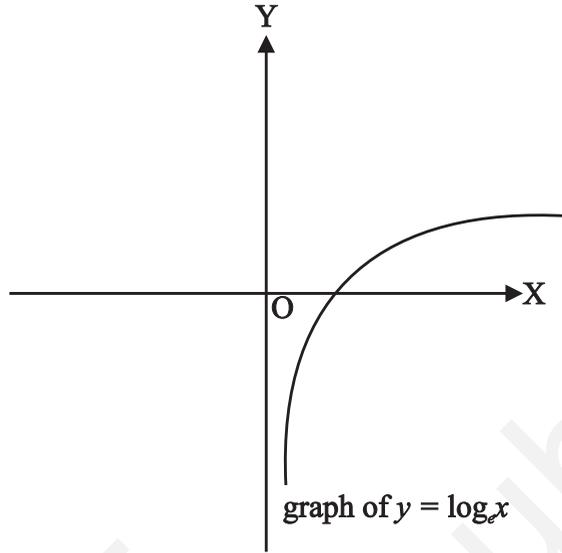


Fig. 13.12

In order to prove the result  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$ , we make use of an inequality involving the expression  $\frac{e^x - 1}{x}$  which runs as follows:

$$\frac{1}{1 + |x|} \leq \frac{e^x - 1}{x} \leq 1 + (e - 2)|x| \text{ holds for all } x \text{ in } [-1, 1] \sim \{0\}.$$

**Theorem 6** Prove that  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$

**Proof** Using above inequality, we get

$$\frac{1}{1 + |x|} \leq \frac{e^x - 1}{x} \leq 1 + |x|(e - 2), x \in [-1, 1] \sim \{0\}$$

Also  $\lim_{x \rightarrow 0} \frac{1}{1 + |x|} = \frac{1}{1 + \lim_{x \rightarrow 0} |x|} = \frac{1}{1 + 0} = 1$

and  $\lim_{x \rightarrow 0} [1 + (e - 2)|x|] = 1 + (e - 2) \lim_{x \rightarrow 0} |x| = 1 + (e - 2)0 = 1$

Therefore, by Sandwich theorem, we get

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

**Theorem 7** Prove that  $\lim_{x \rightarrow 0} \frac{\log_e(1+x)}{x} = 1$

**Proof** Let  $\frac{\log_e(1+x)}{x} = y$ . Then

$$\log_e(1+x) = xy$$

$$\Rightarrow 1+x = e^{xy}$$

$$\Rightarrow \frac{e^{xy} - 1}{x} = 1$$

or  $\frac{e^{xy} - 1}{xy} \cdot y = 1$

$$\Rightarrow \lim_{xy \rightarrow 0} \frac{e^{xy} - 1}{xy} \lim_{x \rightarrow 0} y = 1 \text{ (since } x \rightarrow 0 \text{ gives } xy \rightarrow 0)$$

$$\Rightarrow \lim_{x \rightarrow 0} y = 1 \left( \text{as } \lim_{xy \rightarrow 0} \frac{e^{xy} - 1}{xy} = 1 \right)$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\log_e(1+x)}{x} = 1$$

**Example 5** Compute  $\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x}$

**Solution** We have

$$\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x} = \lim_{3x \rightarrow 0} \frac{e^{3x} - 1}{3x} \cdot 3$$

$$= 3 \left( \lim_{y \rightarrow 0} \frac{e^y - 1}{y} \right), \text{ where } y = 3x$$

$$= 3 \cdot 1 = 3$$

**Example 6** Compute  $\lim_{x \rightarrow 0} \frac{e^x - \sin x - 1}{x}$

**Solution** We have 
$$\lim_{x \rightarrow 0} \frac{e^x - \sin x - 1}{x} = \lim_{x \rightarrow 0} \left[ \frac{e^x - 1}{x} - \frac{\sin x}{x} \right]$$

$$= \lim_{x \rightarrow 0} \frac{e^x - 1}{x} - \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 - 1 = 0$$

**Example 7** Evaluate  $\lim_{x \rightarrow 1} \frac{\log_e x}{x - 1}$

**Solution** Put  $x = 1 + h$ , then as  $x \rightarrow 1 \Rightarrow h \rightarrow 0$ . Therefore,

$$\lim_{x \rightarrow 1} \frac{\log_e x}{x - 1} = \lim_{h \rightarrow 0} \frac{\log_e(1 + h)}{h} = 1 \left( \text{since } \lim_{x \rightarrow 0} \frac{\log_e(1 + x)}{x} = 1 \right).$$

### Exercise 13.2

Evaluate the following limits, if exist

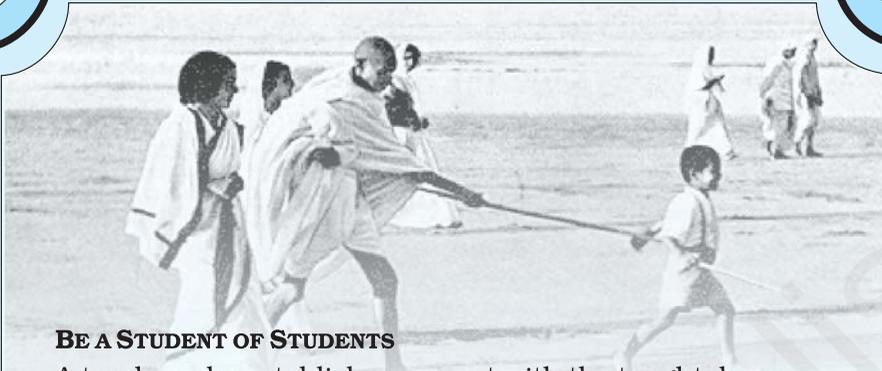
- |  |               |  |               |
|--|---------------|--|---------------|
| 1. $\lim_{x \rightarrow 0} \frac{e^{4x} - 1}{x}$     | (Ans. 4)      | 2. $\lim_{x \rightarrow 0} \frac{e^{2+x} - e^2}{x}$        | (Ans. $e^2$ ) |
| 3. $\lim_{x \rightarrow 5} \frac{e^x - e^5}{x - 5}$  | (Ans. $e^5$ ) | 4. $\lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{x}$       | (Ans. 1)      |
| 5. $\lim_{x \rightarrow 3} \frac{e^x - e^3}{x - 3}$  | (Ans. $e^3$ ) | 6. $\lim_{x \rightarrow 0} \frac{x(e^x - 1)}{1 - \cos x}$  | (Ans. 2)      |
| 7. $\lim_{x \rightarrow 0} \frac{\log_e(1 + 2x)}{x}$ | (Ans. 2)      | 8. $\lim_{x \rightarrow 0} \frac{\log(1 + x^3)}{\sin^3 x}$ | (Ans. 1)      |

## Notes

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## Notes

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### **BE A STUDENT OF STUDENTS**

A teacher who establishes rapport with the taught, becomes one with them, learns more from them than he teaches them. He who learns nothing from his disciples is, in my opinion, worthless. Whenever I talk with someone I learn from him. I take from him more than I give him. In this way, a true teacher regards himself as a student of his students. If you will teach your pupils with this attitude, you will benefit much from them.

*Talk to Khadi Vidyalaya Students, Sevagram  
Harijan Seva, 15 February 1942 (CW 75, p. 269)*

### **USE ALL RESOURCES TO BE CONSTRUCTIVE AND CREATIVE**

What we need is educationists with originality, fired with true zeal, who will think out from day to day what they are going to teach their pupils. The teacher cannot get this knowledge through musty volumes. He has to use his own faculties of observation and thinking and impart his knowledge to the children through his lips, with the help of a craft. This means a revolution in the method of teaching, a revolution in the teachers' outlook. Up till now you have been guided by inspector's reports. You wanted to do what the inspector might like, so that you might get more money yet for your institutions or higher salaries for yourselves. But the new teacher will not care for all that. He will say, 'I have done my duty to my pupil if I have made him a better man and in doing so I have used all my resources. That is enough for me'.

*Harijan, 18 February 1939 (CW 68, pp. 374-75)*